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THE DYNAMICS OF SYSTEMS WITH UNILATERAL CONSTRAINTS*

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The possibility of extending the canonical formalism to a system with ideal unilateral constraints is demonstrated. The problem of the motion of a heavy material point in a vertical plane not below a certain smooth curve is considered as an illustration.

1. We consider a mechanical system M with Lagrange functions of the form

$$L = T - \Pi,\ T = \frac{1}{2}\sum_{i,j=0}^{n} a_{ij}(q_0, \mathbf{q}) q_i^{\cdot} q_j^{\cdot} \tag{1.1}$$
$$\Pi = \Pi(q_0, \mathbf{q}, t),\ \mathbf{q} = (q_1, \ldots, q_n)$$

and an ideal unilateral constraint $q_0 \geqslant 0$. In the intervals between impacts on the constraint, the motion of the system is described by the equations /1/

$$\frac{d}{dt}\frac{\partial L}{\partial q_0^{\cdot}} - \frac{\partial L}{\partial q_0} = F_0,\ \frac{d}{dt}\frac{\partial L}{\partial q_k^{\cdot}} - \frac{\partial L}{\partial q_k} = 0 \quad (k = 1, \ldots, n) \tag{1.2}$$

where F_0 is the reaction of the stressed constraint; $F_0 = 0$ when $q_0 \neq 0$.

Until recently the investigation of systems with unilateral constraints was reduced to examining system (1.2) in finite time intervals between impacts and to "fitting" the boundary conditions at the ends of these intervals (see /2/, for example). The equations of motion of the system M in an arbitrary time interval are obtained in /3/ and the possibilities are examined for applying these equations, which have the Routh form, to the solution of certain problems in mechanics.

The purpose of this paper is to extend the canonical formalism to a system with ideal unilateral constraints, which will enable the developed methods of Hamiltonian mechanics to be used to investigate them.

We will replace the generalized coordinates in the system M by means of the formulas

$$q_0 = Q_0, \quad q_k = \varphi_k(Q_0, \mathbf{Q}), \quad J = \det\left\|\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{Q}}\right\| \neq 0 \tag{1.3}$$
$$\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n),\ \mathbf{Q} = (Q_1, \ldots, Q_n)$$

where we select the functions φ_k so that the relationship $A_{0m} \equiv 0\ (m = 1, \ldots, n)$ is satisfied in the kinetic energy expressions in the new coordinates

$$T = \frac{1}{2}\sum_{i,j=0}^{n} A_{ij}(Q_0, \mathbf{Q}) Q_i^{\cdot} Q_j^{\cdot}$$

Since

$$q_k^{\cdot} = \sum_{m=0}^{n} \frac{\partial \varphi_k}{\partial Q_m} Q_m^{\cdot}$$

these relationships can be written in the form of the equations

$$0 = A_{0m} = \frac{\partial^2 T}{\partial Q_0^{\cdot} \partial Q_m^{\cdot}} = \frac{\partial}{\partial Q_0^{\cdot}}\left(\sum_{j=1}^{n} \frac{\partial T}{\partial q_j^{\cdot}} \cdot \frac{\partial \varphi_j}{\partial Q_m}\right) = \sum_{j=1}^{n}\left(a_{0j} + \sum_{i=1}^{n} a_{ij}\frac{\partial \varphi_i}{\partial Q_0}\right)\frac{\partial \varphi_j}{\partial Q_m}$$

which are a homogeneous set of linear equations in

$$x_j = a_{0j} + \sum_{i=1}^{n} a_{ij}\frac{\partial \varphi_i}{\partial Q_0} \quad (j = 1, \ldots, n)$$

whose determinant is different from zero because of (1.3). Therefore, $x_j = 0\ (j = 1, \ldots, n)$, i.e.

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$$\sum_{i=1}^{n} a_{ij}(Q_0, \varphi) \frac{\partial \varphi_i}{\partial Q_0} = -a_{0j}(Q_0, \varphi) \quad (j = 1, \ldots, n) \tag{1.4}$$

The determinant of the linear system (1.4) in $\partial\varphi_i/\partial Q_0$ is the principal minor of the kinetic energy matrix T; consequently, it is different from zero and system (1.4) is solvable for the derivatives $\partial\varphi_i/\partial Q_0$. Setting

$$\varphi_i \,|_{Q_0=0} = Q_i \quad (i = 1, \ldots, n) \tag{1.5}$$

we formulate the Cauchy problem (1.4), (1.5) for the functions φ_i, in which Q_0 plays the part of the independent variable while Q_i are the initial conditions. Since the Jacobian J equals one for $Q_0 = 0$ because of (1.5), the condition of reversibility of the substitution (1.3) is satisfied at least for sufficiently small values of Q_0.

Furthermore, we shall assume that the substitution of (1.3) into system (1.1) is already completed and $a_{0m} \equiv 0$ $(m = 1, \ldots, n)$. Then the first equation in (1.2) has the following form for $q_0 = q_0^{\cdot} = 0$:

$$a_{00} q_0^{\cdot\cdot} - \frac{\partial L}{\partial q_0}\Big|_{q_0=+0} = F_0 \tag{1.6}$$

Since $a_{00} > 0$, then following /1/, we obtain that

$$F_0 = \max\left\{0, -\frac{\partial L}{\partial q_0}\Big|_{q_0=+0}\right\} \tag{1.7}$$

We determine the auxiliary system M^* by using the Lagrange function $L(q_0, \mathbf{q}, q_0^{\cdot}, \mathbf{q}^{\cdot}, t) = L(|q_0|, \mathbf{q}, q_0^{\cdot}, \mathbf{q}^{\cdot}, t)$ and the generalized force (1.7).

The following relationships

$$q_0(t) = |q_0^*(t)|, \quad \mathbf{q}(t) = \mathbf{q}^*(t) \tag{1.8}$$

are satisfied for the trajectories $\mathbf{Q}(t) = (q_0(t), \mathbf{q}(t))$ and $\mathbf{Q}^*(t) = (q_0(t), \mathbf{q}^*(t))$ of systems M and M^*.

In fact, for $q_0^* \geqslant 0$ we have $L^* = L$ and the trajectories $\mathbf{Q}(t)$ and $\mathbf{Q}^*(t)$ coincide. Because $\partial L^*/\partial q_0^{\cdot}$, $\partial L^*/\partial \mathbf{q}$ and $\partial L^*/\partial \mathbf{q}^{\cdot}$ are even while $\partial L^*/\partial q_0$ is an odd function in q_0, for $q_0^* < 0$ the equations of motion of system M^* go over into (1.2) by means of the substitution $q_0^* \to -q_0$, and the relations (1.8) are also satisfied.

The quantities $\partial T/\partial \mathbf{q}^{\cdot}$ remain continuous on the curve $\mathbf{Q}(t)$ under impact /4/. The kinetic energy T is also continuous because the constrain is ideal. Therefore, because $a_{0m} \equiv 0$, the quantity $|q_0^{\cdot}(t)|$ is also continuous. In turn, the curve $\mathbf{Q}^*(t)$ in an extremal of the action functional

$$\int_{t_1}^{t_2} L^*(\mathbf{Q}^*(t), \mathbf{Q}^{*\cdot}(t), t)\, dt$$

consequently, $q_0^{*\cdot}(t)$ and $\mathbf{q}^{*\cdot}(t)$ are continuous at points of this curve. After impact, the tangential vectors of the trajectories $\mathbf{Q}(t)$ and $\mathbf{Q}^*(t)$ are therefore symmetric relative to the plane $q_0 = 0$ and relations (1.8) remain valid, which proves their correctness during the whole time of the motion.

Setting

$$p_j = \frac{\partial L^*}{\partial q_j^{\cdot}} \; (j = 0, 1, \ldots, n), \qquad H = \sum_{j=0}^{n} p_j q_j^{\cdot} - L^*$$

we write the equations of motion of system M^* in the canonical form

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 0, 1, \ldots, n) \tag{1.9}$$

where it is necessary to set

$$\frac{\partial H}{\partial q_0}\Big|_{q_0=0} = \min\left\{0, \frac{\partial H}{\partial q_0}\Big|_{q_0=+0}\right\} \tag{1.10}$$

in conformity with (1.6) and (1.7).

The canonical system (1.9), (1.10) determines the motion of system (1.1) uniquely in an arbitrary time interval; in particular, the motion for a stressed constraint occurs under the condition

$$dH/dq_0 \,|_{q_0=+0} \geqslant 0$$

2. As an illustration, we consider the problem of the motion of a material point in a vertical plane not below a smooth curve $y = f(x)$. We select the measurement units in such a

way that the mass and weight of the point is unity. We take $q_0 = y - f(x)$, $q_1 = x$ as the generalized coordinates. The Lagrange function has the form

$$L = {}^1/_2 (x^{\cdot 2} + y^{\cdot 2}) - y = {}^1/_2 \{q_1^{\cdot 2} + [f'(q_1) q_1^{\cdot} + q_0^{\cdot}]^2\} - f(q_1) - q_0, \quad q_0 \geqslant 0 \tag{2.1}$$

We make the reducing substitution (1.3): $q_0 = Q_0$, $q_1 = \varphi(Q_0, Q_1)$ where φ is the solution of the Cauchy problem

$$\frac{\partial \varphi}{\partial Q_0} = -G(\varphi), \quad \varphi|_{Q_0=0} = Q_1, \quad G = \frac{f'}{1+f'^2} \tag{2.2}$$

In the new variables the Lagrangean (2.1) takes the form

$$L = \frac{E(\varphi)}{2} Q_0^{\cdot 2} + \frac{\varphi_1^2}{2E(\varphi)} Q_1^{\cdot 2} - f(\varphi) - Q_0, \quad E = \frac{1}{1+f'^2}, \quad \varphi_1 = \frac{\partial \varphi}{\partial Q_1}$$

The equations of motion of the auxiliary system M^* have the canonical form (1.9), (1.10) with Hamilton functions of the form

$$H = \frac{1}{2E(\psi)} P_0^2 + \frac{E(\psi)}{2\psi_1^2} P_1^2 + f(\psi) + |Q_0| \tag{2.3}$$
$$\psi = \varphi(|Q_0|, Q_1), \quad \psi_1 = \frac{\partial \psi}{\partial Q_1}$$

We will examine certain special cases of the motion.

1) For $Q_0 = P_0 = 0$, motion of the point along a curve occurs. Here $\varphi = \psi = Q_1$, $\varphi_1 = \psi_1 = 1$ and the Hamiltonian (2.3) has the form

$$H|_{Q_0=P_0=0} = {}^1/_2 E(Q_1) P_1^2 + f(Q_1)$$

The condition for motion of a point along a curve appears thus:

$$\partial H / \partial Q_0 |_{Q_0=+0} = E[1 + E^2 P_1^2 f''(Q_1)] = E^{1/2} (\cos \alpha + \varkappa v^2) \geqslant 0$$

where $\varkappa$ and α are the curvature of the curve and the angle it makes with the abscissa axis, and v is the magnitude of the point velocity.

2) If $f'(x_0) = 0$, the system allows a motion for which the point periodically jumps up above the curve, while its abscissa is constant and equal to x_0. Corresponding particular solutions of (1.9), (1.10) with Hamiltonian (2.3) have the following form for $-\tau/2 \leqslant t \leqslant \tau/2$

$$Q_0 = {}^1/_2 t [2(2h)^{1/2} - |t|], \; P_0 = (2h)^{1/2} - |t|, \; Q_1 = x_0, \; P_1 = 0 \tag{2.4}$$

where h is the height of the jump upward, and $\tau = 4(2h)^{1/2}$ is the period of the motion under consideration (i.e., the time interval between the k-th and $k+2$-th collisions, $k = 1, 2, \ldots$).

Let us investigate the orbital stability of these periodic motions, i.e., the stability of the oribital parameter h and the variables Q_1, P_1 to perturbations. To do this we pass from the variables Q_0, P_0 over to the "action-angle" variables I, w by means of the formulas

$$Q_0 = 2\left(\frac{3}{2\pi^2} I\right)^{2/3} w(\pi - |w|), \quad P_0 = 2\left(\frac{3}{2\pi^2} I\right)^{1/3} \left(\frac{\pi}{2} - |w|\right) \tag{2.5}$$

for $-\pi \leqslant w \leqslant \pi$; the substitution is 2π-periodic in w.

The solutions (2.4) take the form

$$I = I_0, \quad w = \frac{\pi}{2}\left(\frac{3\pi}{2} I_0\right)^{-1/3} t + w_0, \quad Q_1 = x_0, \quad P_1 = 0$$

We will describe the perturbed motion by means of the variables r, ξ, η, defined by the relationships

$$r = I - I_0, \quad \xi = Q_1 - x_0, \quad \eta = P_1$$

By virtue of (2.5) the variables $|Q_0|$ and P_0 are analytic in r for $I_0 \neq 0$, consequently, the Hamilton function (2.3) will be a smooth function of the perturbations. The presence of the quantity $|Q_0|$ in the equations results solely in the non-differentiability of the Hamiltonian with respect to the angular variable w.

Note that the smoothness mentioned is achieved on the right sides of the perturbed-motion equations because of the introduction of the "action-angle" variables and is therefore due to the canonical form of the equations of motion. For the Routh equations that underlie the method used in /3/, there is no such smoothness relative to the perturbation of the oribtal parameter.

The regular nature of the perturbed motion Hamiltonian enables the stability of the periodic solution to be investigated by known algorithms /5, 6/.

As a result of such an investigation, it was found that the stability domain has the form $0 < k < {}^1/_2$ in a linear approximation, where the quantity $k = {}^1/_2 f''(x_0)({}^3/_2 \pi I_0)^{2/3}$ is the ratio of the height of the upward jump in the periodic solution and the radius of curvature of the curve $y = f(x)$ at the point x_0.

For $k = {}^3/_8$ the characteristic exponents λ_1 and λ_2 of the linearized system of perturbed-motion equations are related to the third order resonance relationship $\lambda_1 = 3\lambda_2$. Calculations show that if $f'''(x_0) \neq 0$ here, then the periodic motion under investigation is unstable.

For the remaining values of k in the interval $(0, {}^1/_2)$ the solution of the stability question depends on the parameters

$$\varkappa_1 = \frac{f'''(x_0)}{[f''(x_0)]^2}, \quad \varkappa_2 = \frac{f^{IV}(x_0)}{[f''(x_0)]^3}$$

For $k \neq {}^1/_4$ (there is no fourth-order resonance $\lambda_1 = 4\lambda_2$), orbital stability of the periodic solutions under consideration hold in the general case of non-degeneracy of the normal form.

In particular, calculations performed for the parabola $(\varkappa_1 = \varkappa_2 = 0)$ and the sinusoid $(\varkappa_1 = 0,\ \varkappa_2 = -1)$ show that the solutions (2.4) are stable for these curves for all values of k in the interval $(0, {}^1/_2)$.

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SOME CONDITIONS FOR THE EXISTENCE AND STABILITY OF PERIODIC OSCILLATIONS IN NON-LINEAR NON-AUTONOMOUS HAMILTONIAN SYSTEMS*

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The sufficient conditions for the existence and uniqueness of periodic solutions are obtained for non-autonomous Hamiltonian systems by the method of continuation with respect to the parameter /1/ (similar results were established for certain vector equations by other methods in /2, 3/). Using the theorem on the directed width of stability regions /4/, stability criteria to a first approximation of these solutions are obtained. The effect of small dissipative forces on stability is investigated. Systems are considered in which some of the generalized coordinates are angular. The conditions for the existence, uniqueness, and stability are obtained, as well as the upper bounds of solutions that correspond to periodic rotational motions of the angular coordinates with any preassigned average velocities that are multiples of the perturbing effect. The periodic oscillatory and rotational motions of two coupled pendulums are considered, as an example.

1. We consider the system

$$x_i^{\cdot} = \frac{\partial H}{\partial x_{i+n}}, \quad \dot{x}_{i+n} = -\frac{\partial H}{\partial x_i}, \quad i = 1, \ldots, n \tag{1.1}$$

where $x_1, \ldots, x_n$ are the generalized coordinates, $x_{n+1}, \ldots, x_{2n}$ are the momenta, and the Hamiltonian function $H(x_1, \ldots, x_{2n}, \omega t)$ is doubly differentiable with respect to x_i and 2π-periodic in ωt.

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